4 Noise in Communication Systems

4.1 Definition of Noise

Noise can be defined as any unwanted signals, random or deterministic, which interfere with the faithful reproduction of the desired signal in a system. Stated another way, any interfering signal, which is usually noticed as random fluctuations in voltage or current tending to obscure and mask the desired signals is known as noise. These unwanted signals arise from a variety of sources and can be classified as man-made or naturally occurring.

Man-made types of interference (noise) can practically arise from any piece of electrical or electronic equipment and include such things as electromagnetic pick-up of other radiating signals, inadequate power supply filtering or alias terms. Man-made sources of noise all have the common property that their effect can be eliminated or at least minimised by careful engineering design and practice.

Interference caused by naturally occurring noise are not controllable in such a direct way and their characteristics can best be described statistically. Natural noise comes from random thermal motion of electrons, atmospheric absorption and cosmic sources.

4.2 Statistical Description of Signals

Since noise is mostly random in nature, it is best described through its statistical properties. In this section the main parameters and in-between relations for noise description are presented and analysed. Without going into details of random variables and stochastic processes, expressions are developed to describe noise through its power spectral density (frequency domain) or equivalently the auto-correlation function (time domain). This section gives a quick review of some elementary principles, a full description can be found in literature [Papoulis, 1984, Proakis, 1989]. It should be noted that the description is valid for both random and deterministic signals.

4.2.1 Time-Averaged Noise Representations

In forming averages of any signal, whether random or deterministic, we find parameters which tell us something about the signal. But much of the detailed information about the signal is lost through this process. In the case of random noise, however, this is the only useful quantity.

Suppose $n(t)$ is a random noise voltage or current. A typical waveform is illustrated in Fig. 4.1. We now define the following statistical quantities of $n(t)$:

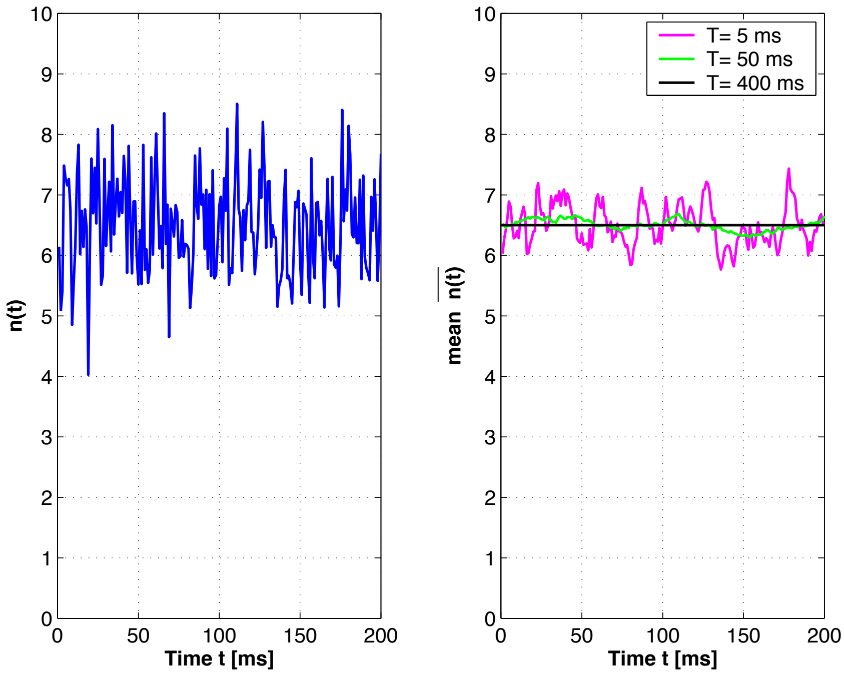

Figure 4.1: A random noise waveform and its average value

Mean Value

The mean value of $n(t)$ will be refered to as $\overline{n(t)}$, η_T or $E\{n(t)\}$. It is given by:

$$\overline{n(t)} = E\{n(t)\} = \eta_T = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)dt. \tag{4.1}$$

where $\overline{n(t)}$ is often referred to as dc or *average* value of $n(t)$. For practical calculation of the mean value, the averaging time T has to be chosen large enough to smooth the fluctuations of $n(t)$ adequately. Fig. 4.1 shows the averages $\overline{n(t)}$ calculated by sliding a window centred at t and extending from $t - T/2$ to $t + T/2$ over $n(t)$. It is seen that for small averaging time $T = 5$ ms there is still a considerable amount of fluctuation present whereas for $T = 400$ ms the window covers the whole time signal which results in a constant average value.

Mean-Square Value

$$\overline{n^2(t)} = E\{n^2(t)\} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2 dt. \tag{4.2}$$

Aside from a scaling factor the mean-square value $\overline{n^2(t)}$ in (4.2) gives the time averaged power P of $n(t)$. Assuming $n(t)$ to be the noise voltage or current, the scaling factor will be equivalent to a resistance, which is often set equal to $1\,\Omega$. The square root of $\overline{n^2(t)}$ is known as the *root-mean-square* (rms) value of $n(t)$. The advantage of the rms notation is that the units of $\sqrt{\overline{n^2(t)}}$ are the same as those for $n(t)$.

AC Component

The ac or fluctuation component $\sigma(t)$ of $n(t)$ is that component that remains after "removing" the mean value $\overline{n(t)}$ and is defined as (see also Fig. 4.2):

$$\sigma(t) = n(t) - \overline{n(t)}. \tag{4.3}$$

Variance

The variance $\overline{\sigma^2(t)}$ defined by:

$$\overline{\sigma^2(t)} = E\{\sigma^2(t)\} = E\{(n(t) - \overline{n(t)})^2\} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\sigma(t)|^2 dt \tag{4.4}$$

is equal to the power of the ac component of $n(t)$ (aside from a resistance scaling factor). This can be showed by substituting (4.3) into (4.2) giving:

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\overline{n(t)} + \sigma(t)|^2 dt. \tag{4.5}$$

Using the fact that $\overline{n(t)}$ is a constant and that the mean of $\sigma(t)$ is zero by definition, we get:

$$\overline{n^2(t)} = \left|\overline{n(t)}\right|^2 + \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |\sigma(t)|^2 dt. \tag{4.6}$$

In the above equation, the time averaged power of $n(t)$ is written as the power sum of the dc and ac signal components. The variance is a measure of how strong the signal is fluctuation about the mean value.

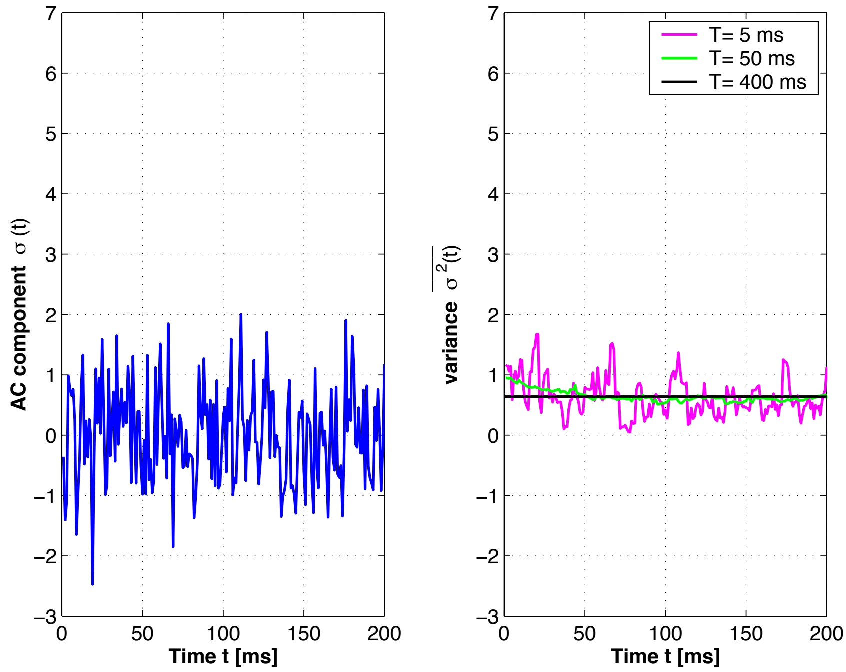

Figure 4.2: AC component and variance of random noise waveform

Validity of Time-Based Statistics

Mathematically noise is modeled as a stochastic process, where the noise can be considered as the realization of a random variable. Consider the stochastic process $x(t)$, at time instance t_1, $x(t_1)$ is a randome variable; its expected value (mean) is $\eta(t_1) = E\{x(t_1)\}$, and can be estimated by observing N samples (i.e. realizations) $x(t_1, \xi_i)$ through:

$$\hat{\eta}(t_1) = \frac{1}{N} \sum_i x(t_1, \xi_i) \tag{4.7}$$

where it can be shown that $\hat{\eta}(t_1)$ is a consistent estimator of $\eta(t_1)$. However, the above expression can only be used if a large number of realizations $x(t_1, \xi_i)$ of $x(t_1)$ are available. In many applications this is not the case and mostly only a single sample (realization) of $x(t, \xi_1)$ is available. We therefore estimate $\eta(t)$ in terms of the time average of the given sample. For this to be possible, first, $E\{x(t)\}$ should *not* depend on t; in this case $x(t)$ is called a stationary process. To arrive at the second condition, we form the time average according to:

$$\eta_T = \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt. \tag{4.8}$$

which is a random variable with mean:

$$E\{\eta_T\} = \frac{1}{T} \int_{-T/2}^{T/2} E\{x(t)\} = \eta. \tag{4.9}$$

Now if the time average η_T computed from a single realization of $x(t)$ tends to the ensemble average η as $T \to \infty$ then the random process is called mean-ergodic.

As such, for ergodic random processes a single time realization can be used to obtain the moments of the process. Thus, the expressions in (4.1) to (4.6) are only correct if the stochastic process is both stationary and ergodic [Papoulis, 1984, Proakis, 1989]. In addition, since the measuring time T is finite the quantities are only *estimated* values of the moments. In practice, the only quantity accessible to measurements is $n(t)$, which forces us to assume a stationary, ergodic stochastic process.

Drill Problem 33 *Calculate the (a) average value, (b) ac component, and (c) rms value of the periodic waveform* $v(t) = 1 + 3\cos(2\pi f t)$.

Drill Problem 34 *A voltage source generating the waveform of drill problem 33 is connected to a resistor* $R = 6\,\Omega$. *What is the power dissipated in the resistor?*

4.2.2 Fourier Transform

The definition of the Fourier transform [Stremler, 1982] is given by

$$F(f) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi f t} dt \tag{4.10}$$

and the inverse Fourier transform

$$f(t) = \mathcal{F}^{-1}\{F(f)\} = \int_{-\infty}^{\infty} F(f) e^{j2\pi f t} df. \tag{4.11}$$

If the signal $f(t)$ is a *power signal*, i.e., a signal with finite power but infinite energy, the integral in (4.10) will diverge. However, considering the practical case of a finite observation time T and assume that the signal is zero outside this interval, is equivalent to multiplying the signal by the unit gate function rect(t/T). In this case the Fourier transform can be written as:

$$F_T(f) = \mathcal{F}\{f(t)\mathrm{rect}(t/T)\} = \int_{-T/2}^{T/2} f(t) e^{-j2\pi f t} dt \tag{4.12}$$

Note that the multiplication by the rect-function in the time domain is equivalent to a convolution by a sinc-function in the frequency domain.

4.2.3 Correlation Functions

In the following, two statistical functions are introduced which can be used to investigate the similarity between random functions.

Auto-Correlation Function

The *auto-correlation* function $E\{f^*(t)f(t+\tau)\} = R_f(\tau)$ of a signal $f(t)$ is defined as [Stremler, 1982]:

$$E\{f^*(t)f(t+\tau)\} = R_f(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t)f(t+\tau)dt \tag{4.13}$$

where $f^*(t)$ is the complex conjugate of $f(t)$. Note that the subscript $_f$ is added to the autocorrelation funciton $R(\tau)$ to indicate the random variable or function that is considered. The auto-correlation function (4.13) is often used in signal analysis, it gives a similarity measure of the signal $f(t)$ with itself versus a relative time shift by an amount τ. For slowly varying time signals, the signal values doesn't change rapidly over time which will result in a flat auto-correlation function $R_f(\tau)$. Noise signals on the other hand, tend to have rapid fluctuations giving rise to an auto-correlation function with a sharp peak for $\tau = 0$ (no time shift) and quickly falling to zero for increasing τ. As an example Figs. 4.3 and 4.4 show the time signals and the corresponding auto-correlation functions for both an exponential and a random noise signal.

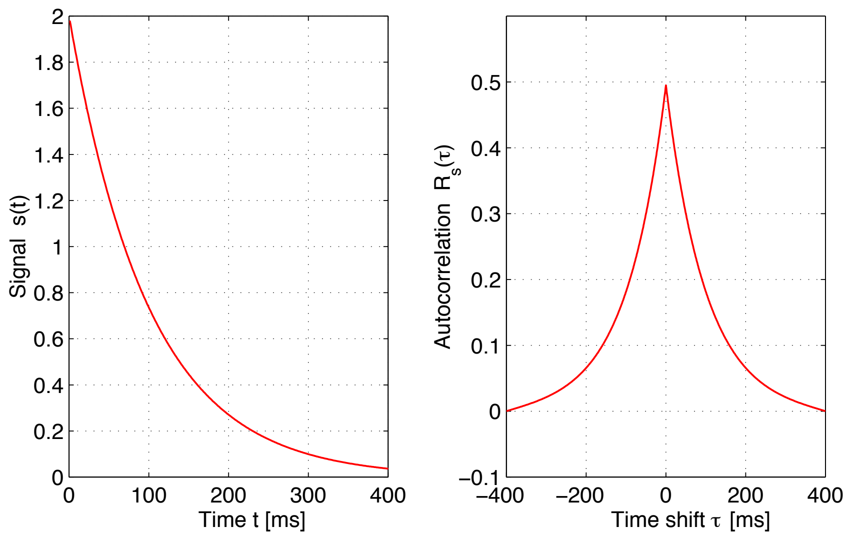

Figure 4.3: Time signal and auto-correlation function of exponential waveform

When dealing with random variables, the auto-correlation function $R_f(\tau)$ is a statistical quantity describing the stochastic process. Note that setting $\tau = 0$ in (4.13) yields $R_f(0) = E\{f^2(t)\} = \overline{f^2(t)}$ which is the average power of the signal as is readily seen by comparing to (4.2). Again, we note that the definition of $R_f(\tau)$ as given by (4.13) is only valid if the stochastic process is both stationary and ergodic.

It can be shown, that taking the Fourier transform (with respect to τ) of both sides of (4.13) yields [Proakis, 1989]:

$$\mathcal{F}_\tau\{R_f(\tau)\} = \lim_{T\to\infty} \frac{1}{T} F_T^*(f)F_T(f) = \lim_{T\to\infty} \frac{1}{T} |F_T(f)|^2 \tag{4.14}$$

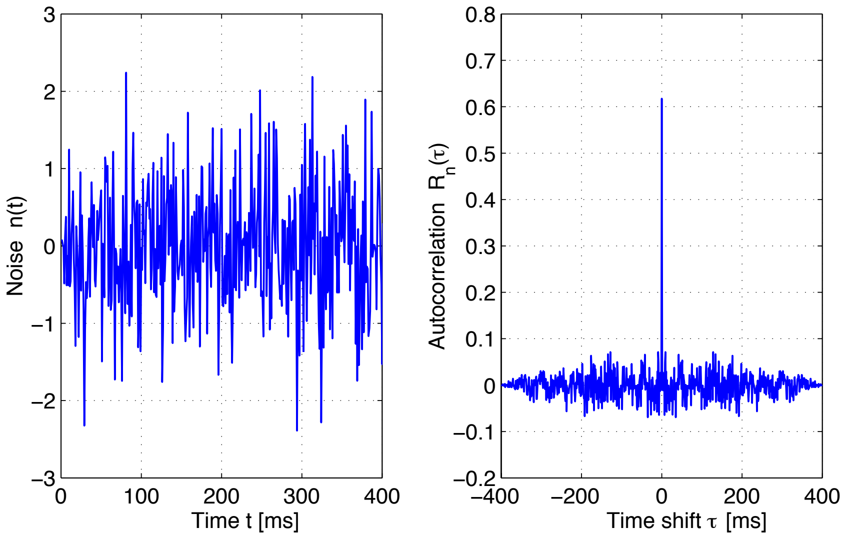

Figure 4.4: Time signal and auto-correlation function of random noise waveform

Thus we arrive at the important conclusion, that the correlation integral resulting in $R_f(\tau)$ corresponds to a multiplication in frequency domain. Or, equivalently we can say that instead of evaluating the integral in (4.13) we can calculate the Fourier transform of $f(t)$ according to (4.12), determine $|F_T(f)|^2$ and use the inverse Fourier transform to get $R_f(\tau)$. The limit for $T \to \infty$ in (4.14) mainly reminds us of the finite observation time, practically this limit means that we have to observe the signal for a sufficient time period.

Cross-Correlation Function

A function closely related to the auto-correlation function, which can be used as a similarity measure between two different signals, is the cross-correlation function. For two waveforms $f(t)$ and $g(t)$, the cross-correlation function $E\{f^*(t)g(t+\tau)\} = C_{fg}(\tau)$ is defined as

$$E\{f^*(t)g(t+\tau)\} = C_{fg}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} f^*(t)g(t+\tau)dt \tag{4.15}$$

The auto-correlation is considered a special case of the cross-correlation, since it can be obtained by setting $f(t) = g(t)$ in (4.15). The interpretation of the cross-correlation between two signals is similar to that of the auto-correlation, with the advantage that additional information, such as the time shift between two similar signals can be deduced.

In general, it can be shown that when *uncorrelated* signals are added, the *average power of the sum* is equal to the *sum of the average powers* of the signals.

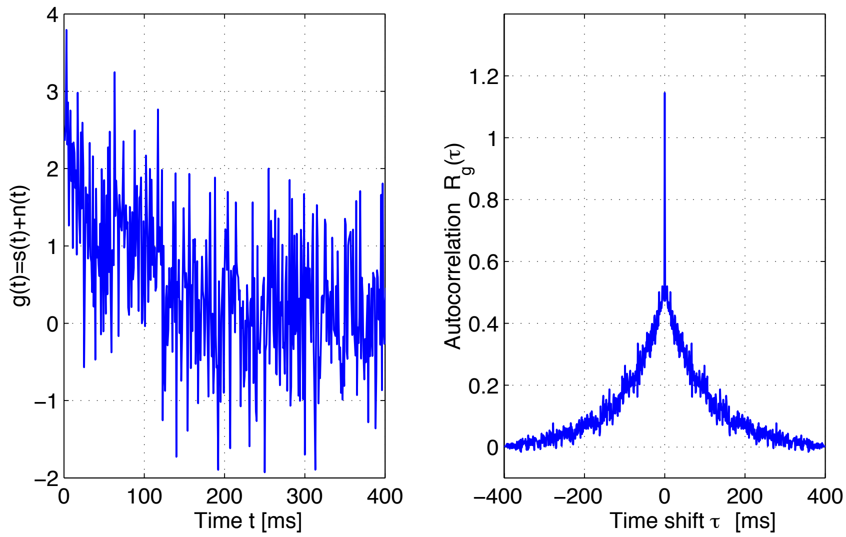

Figure 4.5: Time signal and auto-correlation function of exponential waveform plus noise

A useful application of the correlation function, is the detection of signals masked by additive noise. Fig. 4.5 shows the time waveform $g(t)$ which results when the random noise $n(t)$ from Fig. 4.4 is added to the signal $s(t)$ of Fig. 4.3. It is seen that the signal waveform is immersed in a considerable amount of noise and it would be difficult to detect $s(t)$ out of $g(t)$. However, the auto-correlation function $R_s(\tau)$ can clearly be recognized from the auto-correlation of the summed signals $R_{s+n}(\tau)$. The detection of signals masked by additive noise is a main issue when dealing with communication systems.

Drill Problem 35 *Consider two functions $f(t) = \sin(2\pi ft)$ and $g(t) = \sin(2\pi ft + \theta)$. Find the expression for the cross-correlation function $C_{fg}(\tau)$ of the two functions. Compute the value of $C_{fg}(\tau)$ for $\theta = 0,\ \pi/4$, and $\pi/2\,\text{rad}$. Note that the first case represents the auto-correlation of $\sin x$, and the last is the cross-correlation of $\sin x$ with $\cos x$.*

Drill Problem 36 *Two voltage sources $v_1(t)$ and $v_2(t)$ are connected in series such that the resulting voltage $v_s(t) = v_1(t) + v_2(t)$. Calculate the total power of $v_s(t)$ ($1\,\Omega$ scaling) assuming the signals of the two voltage sources to be completely uncorrelated with each other, i.e. $R_{v_1v_2}(\tau) = R_{v_2v_1}(\tau) \equiv 0$. The signals $v_1(t)$ and $v_2(t)$ are assumed to have rms voltages of $3\,\text{V}$ and $5\,\text{V}$ respectively.*

4.2.4 Power Spectral Density

In the following, we will be dealing with truncated signals, i.e. the signals are only considered within a finite time interval $[-T/2, T/2]$ thus assuming the signal to be zero outside this interval. The mathematical representation is not as strait forward as for infinite time signals, however, practical consideration show that this extra effort is needed. Parseval's theorem for truncated signals state that:

$$\int_{-T/2}^{T/2} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F_T(f)|^2 df. \tag{4.16}$$

Noting the similarity between the first term in (4.16) and the time averaged power P of a signal as given in (4.2) we write:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |F_T(f)|^2 df. \tag{4.17}$$

The first integral in the equation above is easy to understand, it shows that in order to obtain the total power of a signal we must add together the power contribution of each time increment, which is done through the integration over time t. Whether we are dealing with voltage or current is indifferent if we assume a $1\,\Omega$ resistance. We know, that evaluating the integral over a finite time period, will give us the signal power within this time. The first integral is thus also valid over each time interval. However, Parseval's equation suggests a second procedure to calculate the total power, which is performed in the frequency domain. The last term in (4.17) shows that the summation of $|F(f)|^2$ over all frequencies f will also result in the total power P. Defining a *power spectral density function* $S(f)$ in units of Watts per Hz such that its integral over frequency is equal to the total power, gives:

$$\int_{-\infty}^{\infty} S(f) df = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |F_T(f)|^2 df. \tag{4.18}$$

In addition, we insist that $S(f)$ also gives the power over each frequency increment, which means that the integration of the power density function over a frequency range Δf will give the total power for this frequency interval. It can be shown, that under certain conditions –which are fulfilled for most practical signals of interest– $S(f)$ is related to $|F_T(f)|^2$ through:

$$S(f) = \lim_{T\to\infty} \frac{|F_T(f)|^2}{T}. \tag{4.19}$$

Through (4.14) the relation between the power spectral density and the Fourier transform of the auto-correlation function is given:

$$S(f) = \mathcal{F}\{R_f(\tau)\} \tag{4.20}$$

When evaluating the distribution of noise power over frequency, the power spectral density $S(f)$ should be the function to examine, rather then the Fourier transform. The reason for this, is, that the Fourier transform of a random quantity (cf. the expression in (4.12)) is also a random quantity, which in this sense does not give us any useful information. As we know, for random signals we need to investigate the statistical properties. Thus in case we are interested in the frequency content of noise, we compute the Fourier transform of the auto-correlation function as given in the expression above.

Using (4.17), (4.18) and (4.13) we get:

$$P = \overline{f^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \int_{-\infty}^{\infty} S(f) df = R_f(0) \tag{4.21}$$

The above expression can be used to calculate the total power using either the time domain signal, the power spectral density function, or the auto-correlation function. Care must be taken, when the resistive scaling factor is not equal to $1\,\Omega$.

4.3 Noise in Linear Systems

When designing and characterizing communication systems, noise is an important parameter which must be accounted for. In general, the different physical noise sources in addition to other man-made noise sources contribute to the total noise in the system. In the following, noise in linear time invariant (LTI) systems is investigated.

4.3.1 Band Limited White Noise

The power spectral density shall be used to describe noise. Knowing that random noise tends to have rapid fluctuations, we assume a noise voltage $n(t)$ having the auto-correlation function:

$$R_n(\tau) = \frac{N_o}{2}\delta(\tau) \tag{4.22}$$

where $\delta(\tau)$ is the impulse function. Thus, $R_n(\tau)$ is zero for all $\tau \neq 0$, which indicates completely uncorrelated noise signal except for zero time shift. Taking the Fourier transform of $R_n(\tau)$ the power spectral density is:

$$S_n(f) = \mathcal{F}\{R_n(\tau)\} = N_o/2 \quad [\text{Watt/Hz}] \tag{4.23}$$

The power spectral density is constant for all frequencies, thus it contains all frequency components with equal power weighting. This type of noise is designated as *white noise* in analogy to white light. The factor of one-half in (4.23) is necessary to have a two-sided power spectral density.

A problem arises when we try to calculate the total power of white noise, since:

$$P_n = \int_{-\infty}^{\infty} \frac{N_o}{2} df \quad \rightarrow \quad \infty \tag{4.24}$$

which implies an infinite amount of power and thus cannot be used to describe any physical process.

However, it turns out to be a good model for many cases in which the bandwidth is limited through the system. In this case the power spectral density can be assumed flat within the finite measuring bandwidth, which will restrict the total noise power. What we are dealing with in this case is *band-limited white noise* which will appear as white noise to the measuring system.

The power of band-limited white noise is independent of the choice of operating frequency f_0. If $n(t)$ is zero-mean white noise with the power spectral density equal to $N_o/2$ Watts per Hz, then across a bandwidth B the noise power is

$$P_n = \int_{f_0-B/2}^{f_0+B/2} \frac{N_0}{2} df + \int_{-f_0-B/2}^{-f_0+B/2} \frac{N_0}{2} df = 2\int_{-B/2}^{B/2} \frac{N_0}{2} df = BN_o \quad \text{Watt} \tag{4.25}$$

4.3.2 Transmission of Noise Through an LTI System

The transformation of an input signal $x(t)$ through a linear time invariant (LTI) system is described in the time domain through the *convolution* integral:

$$y(t) = \int_{-\infty}^{\infty} x(t)h(t-\tau)d\tau \tag{4.26}$$

where $y(t)$ is the output signal and $h(t)$ is the impulse response of the LTI system. If the input signal is random, what we are interested in is the power spectral density $S_y(f)$ of the output signal. Substituting (4.26) into (4.13) and performing a transformation of variables we obtain the auto-correlation function of the output signal [Proakis, 1989]:

$$R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_x(\tau+\alpha-\beta)h^*(\alpha)h(\beta)d\alpha d\beta \tag{4.27}$$

from which $S_y(f)$ is obtained through (4.20):

$$S_y(f) = \mathcal{F}\{R_y(\tau)\} = S_x(f)|H(f)|^2 \tag{4.28}$$

Thus, we have the important result that the power density spectrum of the output signal is the product of the input power density spectrum multiplied by the magnitude squared of the frequency transfer function. If the auto-correlation function is desired, it is usually easier to compute the power density spectrum through (4.28) and than perform the inverse transform:

$$R_y(\tau) = \mathcal{F}^{-1}\{S_y(f)\} = \mathcal{F}^{-1}\{S_x(f)|H(f)|^2\} \tag{4.29}$$

If the random input signal is white noise $n_i(t)$ with a power spectral density $N_o/2$, then (4.28) becomes:

$$S_{out}(f) = \frac{N_o}{2}|H(f)|^2 \tag{4.30}$$

Drill Problem 37 *A white noise voltage of power spectral density $S_{in}(f) = N_0/2$ is fed to the lowpass filter illustrated in Fig. 4.6. For the output noise, determine the expression for (a) the power spectral density, (b) the autocorrelation function, and (c) the total power.*

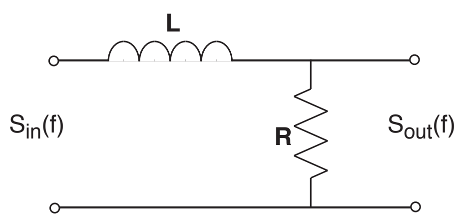

Figure 4.6: RL lowpass filter

4.3.3 Equivalent Noise Bandwidth

The total noise output power from a system with known frequency transfer function $|H(f)|$ can be calculated using (4.28) and (4.21). If the input noise is white, this becomes:

$$P_{out} = \int_{-\infty}^{\infty} S_{out}(f)df = N_o \int_0^{\infty} |H(f)|^2 df. \tag{4.31}$$

The integral is a constant for a given system frequency transfer function. We would like to have a simple expression similar to (4.25) for the output noise power. A reasonable approach would be to define an *equivalent noise bandwidth* B_N of an ideal filter such that the output noise power from the ideal filter and the real system are equal. As shown in Fig. 4.7, we assume that the ideal filters frequency transfer function is flat and equal to $H(f_o)$ within the bandwidth B_N around the centre frequency f_o and zero otherwise.

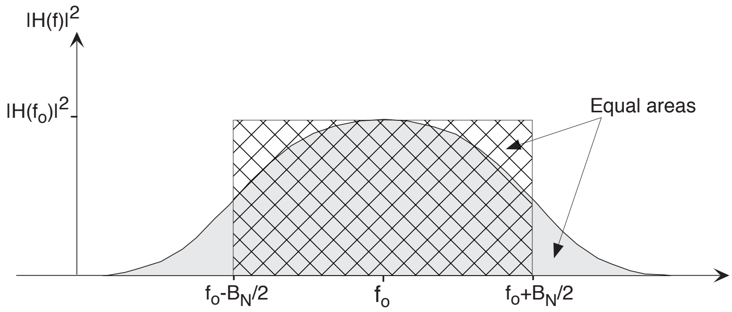

Figure 4.7: A graphical definition of the equivalent noise bandwidth

Thus, the output noise power of the ideal filter is:

$$\begin{aligned} P_{out} &= N_o \int_{f_o-B_N/2}^{f_o+B_N/2} |H(f_o)|^2 df \\ &= N_o|H(f_o)|^2\Big(f_o + B_N/2 - f_o + B_N/2\Big) \\ &= N_o|H(f_o)|^2 B_N. \end{aligned} \tag{4.32}$$

Equating the right hand sides of (4.32) and (4.31), we have:

$$B_N = \frac{\int_0^\infty |H(f)|^2 df}{|H(f_o)|^2} \qquad [\text{Hz}]. \tag{4.33}$$

This definition of the equivalent noise bandwidth B_N allows us to discuss practical linear systems by using their idealized equivalents.

Drill Problem 38 *Compute the equivalent noise bandwidth and the 3-dB bandwidth of the lowpass filter of drill problem 37 with $R = 30\,\Omega$ and $L = 25\,\text{mH}$. Then compute the output noise power for $S_{in}(f) = N_0/2 = 20 \times 10^{-3}\,\text{W Hz}^{-1}$.*

4.3.4 Signal-to-Noise Ratio

Let the input signal power for a given device be $\overline{s_{in}^2(t)}$ and let the input noise power of the device be $\overline{n_{in}^2(t)}$. The input signal-to-noise ratio SNR_{in} is defined as the ratio of the total available signal power to the total available noise power at the input and is given by

$$SNR_{in} = \frac{\overline{s_{in}^2(t)}}{\overline{n_{in}^2(t)}} \tag{4.34}$$

Thus the SNR as defined above gives an indication of the amount of noise power relative to the signal power. Clearly the signal-to-noise ratio at the output of the device is analog to the above expression. Also the definition of the signal-to-noise ratio is independent of the noise source and type.

The SNR is a power ratio which is most often expressed in Decibels:

$$SNR^{\mathrm{dB}} = 10\log_{10}(SNR) \tag{4.35}$$

Thus an $SNR = 13\,\mathrm{dB}$ means that the signal power is twenty times higher than the noise power, while $SNR = 0\,\mathrm{dB}$ means equal signal and noise power.

Drill Problem 39 *An amplifier has an input SNR of $12\,\mathrm{dB}$. Calculate the noise power at the input if the signal power is $-40\,\mathrm{dBm}$.*

Drill Problem 40 *A signal $6\cos(2\pi ft)$ V with $f = 200\,\mathrm{Hz}$ is fed to the input of the filter in drill problem 37. Taking the values of drill problem 38 compute the signal-to-noise ratio at output of the filter.*

4.4 Naturally Occurring Noise

Natural radio noise in telecommunication systems is both picked up by the antenna as well as generated within the system itself. The first effect can be accounted for by the contribution which it makes to the antenna noise temperature. Attenuation due to water vapor and oxygen, clouds and precipitation is accompanied by thermal noise, lightening and other atmospherics which further degrades the applicable signal-to-noise ratio. In addition, extraterrestrial noise of thermal or non-thermal origin may be picked up by the receiving antenna.

This section gives an overview of the different types of naturally occurring noise and defines appropriate quantities for modelling the effect of this noise. To start with, we state Planck's radiation law, which is the the basis for other types of noise.

Planck's Law

In 1900, Max Planck found the law that governs the emission of electromagnetic radiation from a black body in thermal equilibrium [Planck, 1900]. A black body is simply defined as an idealized, perfectly opaque material that absorbs all the incident radiation at all frequencies, reflecting none. A body in thermodynamic equilibrium emits to its environment the same amount of energy it absorbs from its environment. Hence, in addition to being a perfect absorber, a blackbody also is a perfect emitter. The essential point of Planck's derivation is that energy can only be exchanged in discrete portions or quanta equal to hf, where h is Planck's constant h = 6.626×10^{-34} J s and f is the frequency in Hertz.

Then, the energy of the ground level (or state) is 0, of the first level hf, of the second level $2hf$ and so on. In general:

$$E_v = n \cdot hf \quad \text{for} \quad v = 0, 1, 2 \ldots \tag{4.36}$$

where v is the level or state number. Given the number N_v of energy quanta (in Planck's publications these are referred to as *energy elements*) occupying level v results in an energy of vN_vhf for that level. The total energy is obtained by summing up over all states, thus

$$E_{tot} = N_0 \cdot 0 + N_1 hf + N_2 \cdot 2hf + N_3 \cdot 3hf + \ldots \tag{4.37}$$

Now according to quantum mechanics, the probability of occupying an energy level goes down with $e^{-\Delta E/\mathrm{k}T}$ where k = 1.38×10^{-23} J K^{-1} is the Boltzmann constant, T is the absolute temperature in Kelvin, and ΔE the excess energy. Then, the number of energy quanta N_1 at the first level is given by the number at ground state N_0 multiplied by the probability $e^{-hf/\mathrm{k}T}$. Similarly $N_2 = N_1 e^{-hf/\mathrm{k}T} = N_0 e^{-2hf/\mathrm{k}T}$ and so on. The total number of quanta is :

$$N_{tot} = N_0 \left(1 + e^{-hf/kT} + e^{-2hf/kT} + e^{-3hf/kT} + \ldots\right) \tag{4.38}$$

To determine the average energy, we divide the total energy by the total number of energy quanta. The expression can be simplified to give:

$$\overline{E(f)} = \frac{hf}{e^{hf/kT} - 1} \tag{4.39}$$

Using the density of modes we find Planck's law for the black body radiation. Expressed in terms of the brightness of the radiated energy from a blackbody this is given by:

$$B_f(f) = \frac{2hf^3}{c^2} \frac{1}{e^{hf/kT} - 1} \tag{4.40}$$

4.4.1 Thermal Radiation

Thermal radiation is system inherent and is generated through the random thermal motion of electrons in a conducting medium such as a resistor. The path of each electron is randomly oriented due to interaction with other electrons. The net effect of the electron motion is a random current flowing in the conduction medium with an average value of zero. The power spectral density of thermal noise is given by Planck's distribution law (4.39). For the normal range of Temperatures and frequencies well below the optical range the parameter hf/kT is very small, so that $e^{hf/kT} \approx 1 + hf/kT$, and (4.39) can be approximated by:

$$S_n(f) = kT \tag{4.41}$$

The power spectral density as given by (4.41) is independent of frequency and hence is referred to as *white noise spectrum*. Within the bandwidth B the available noise power then is

$$P_n = kTB \tag{4.42}$$

The above expression shows, that if the bandwidth is fixed it is sufficient to know the temperature in order to be able to compute the noise power. This is the reason, why it is common to speak of the noise temperature when referring to the noise power (even if the noise source is not thermal).

For T =300 K, i.e. at room temperature, we get a noise power of $N_T = -114$ dBm per MHz bandwidth. It is worth remembering this number as a reference and using it to compute the approximate noise power for a given bandwidth. For example the noise power for a 20 MHz system would be $N_T = -101$ dBm.

Knowing the available power to the network, we want to define the cirucit equivalent of the noisy resistor. This is done by considering a voltage source of rms voltage V_n connected in series with the resistor R.

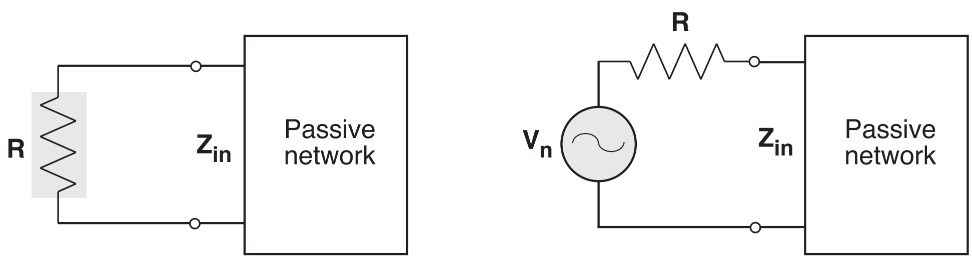

Figure 4.8: Noisy resistor connected to a network (left) and its equivalent circuit (right)

The noise power delivered to a network of input impedance Z_{in} is:

$$P_n = \left| \frac{V_n}{R + Z_{in}} \right|^2 R_{in} \tag{4.43}$$

where R_{in} is the resistive component of Z_{in}. If $Z_{in} = R$, which is the condition for maximum power transfer, we find that

$$P_n = \frac{V_n^2}{4R} \tag{4.44}$$

Substituting for P_n from (4.42) and solving for V_n gives

$$V_n = \sqrt{\overline{v_n^2(t)}} = 2\sqrt{kTRB} \tag{4.45}$$

4.4.2 Extraterrestrial Noise

Space is the source of mostly broadband noise which can be considered as plane electromagnetic waves. Cosmic radiation has to be accounted for if either the main lobe or the sidelobes of the receiving antenna are directed towards space. The noise sources are both thermal and non-thermal emission from the Sun, the Moon, the Cassiopeia and planets and from elsewhere in our galaxy and other galaxies.

If the emission is of thermal origin its contribution to noise power can be described through the spectral brightness as given in (4.40) which is the power density in Watt per unit solid angle per unit area per Hertz. At radio-frequencies where $hf \ll kT$ the spectral brightness B_f is given by the Rayleigh-Jeans law:

$$B_f = \frac{2kT_c}{\lambda^2} \qquad \text{in} \quad \left[\frac{\text{Watt}}{\text{m}^2 \cdot \text{sr} \cdot \text{Hertz}}\right] \tag{4.46}$$

where
T_c is the brightness temperature,
λ is the wavelength and
k is Boltzmann's constant

The actual noise power received within a narrow frequency range depends on the direction of the main lobe and the side lobes of the receiving antenna and on the effective area of the antenna. Thus in general the spectral brightness of an extended source is a function of the direction relative to the antenna coordinates. For discrete sources (such as the Sun), which lie within the main lobe of the antenna and subtend a solid angle Ω_s that is much smaller than the antenna main-beam solid angle, the spectral power density becomes

$$p = \frac{2kT_c}{\lambda^2}\Omega_s \quad \text{W m}^{-2}\,\text{Hz} \tag{4.47}$$

Further use of the spectral brightness will be made in a later section when the total noise power percepted by an antenna will be evaluated in detail.

In the general case B_f varies as λ^n where n is known as the spectral index. Thus for the thermal emission of a black body $n = -2$. For non-thermal emission (4.46) can still be used but the brightness temperature T_c is no longer related to the thermal emission but is an equivalent brightness temperature, in addition the spectral index has to be specified.

Background Radiation: The entire Universe is saturated with what is known as microwave background radiation, a remnant of the Big Bang. After the Big Bang, the formation of matter, space and time out of virtually nothing, the prevailing temperatures were at first almost inconceivably high. However, as the Universe expanded the temperature sank to approximately $-270\,°\mathrm{C}$, the temperature that it is today. The expansion of space lengthened the wavelength of the electromagnetic radiation until it entered the microwave range. Today, this radiation can be measured reaching us evenly from all directions of space, thus the term “background radiation”. It would “heat up” any colder object to the space temperature of $3\,\mathrm{K}$ (note that absolute zero Kelvin is $-273\,°\mathrm{C}$)

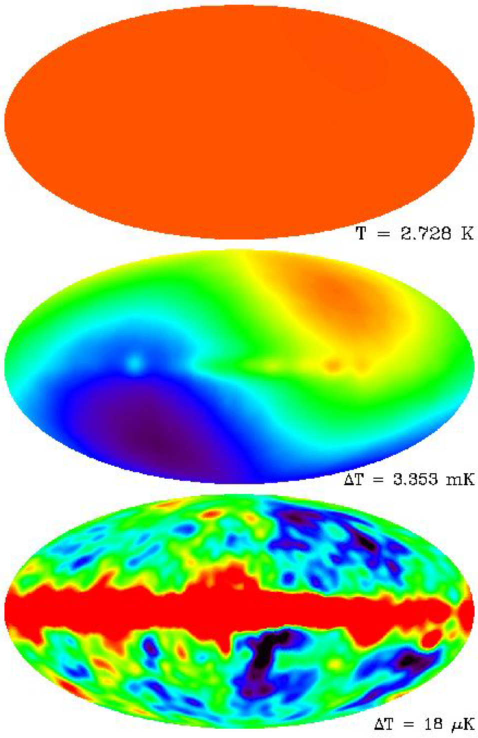

Figure 4.9: Temperature of the cosmic microwave background radiation as determined with the COBE satellite during the first two years of observation. The plane of the Milky Way Galaxy is horizontal across the middle of each picture. The temperature range is 0 – 4K for the top, $3.3\,\mathrm{mK}$ for the middle, and $18\,\mu\mathrm{K}$ for the bottom image respectively.

4.4.3 Absorption Noise

When energy is absorbed by a body the same energy is reradiated as noise as shown by the theory of black body emission. Otherwise the temperature of some bodies would rise and that of others fall. In the case of a radiating antenna the energy is partially absorbed by the atmosphere and reradiated as noise. The effective absorption noise temperature T_{ab} given as a function of the ambient temperature T_a and the attenuation L_a is:

$$T_{ab} = T_a(L_a - 1) \tag{4.48}$$

Note that T_{ab} is not identical to the physical (ambient) temperature of the atmosphere and increases with increasing atmospheric attenuation. Table 4.1 shows some values for L_a and T_{ab} when the ambient temperature T_a is 300K.

L_a [dB]	0	1	3	10
L_a (power ratio)	1	1.26	2	10
T_{ab} [K]	0	78	300	2700

Table 4.1: Absorption noise temperature T_{ab} for different values of attenuation and an ambient temperature $T_a = 300$K

The Attenuation of the atmosphere depends strongly on frequency. Water vapour and oxygen cause a high atmospheric attenuation in the 20GHz and 60GHz frequency bands. The frequency band below 10GHz exhibits the lowest noise temperature.

4.4.4 Additional Natural Noise Sources

Two additional types of noise should be mentioned here, these are:

Shot noise: this type of noise occurs when the quantisation of electrical charge carrier become manifest. It arises in physical devices when a charged particle moves through a potential gradient without collision and with a random starting time. This is the case in vacuum tubes due to the random emission of electrons from the cathode and in many semiconductor components as a result of the diffusion of minority carriers and the random generation and recombination of electron-hole pairs. For these cases the power spectral density is approximately flat up to frequencies in the order of $1/\tau$, where τ is the transit time or lifetime of the charge carriers. In terms of the mean current, the power spectral density is

$$S_{shot} = q\overline{i(t)} + 2\pi\overline{i(t)}^2\delta(f) \tag{4.49}$$

where q is the charge of an electron $= 1.6 \cdot 10^{-19}$ coulomb. The first term in (4.49) corresponds to the ac or fluctuation part of the noise current and the second term corresponds to the nonzero mean value.

1/f noise: lots of components exhibit $1/f$ noise which appears at low frequencies (depending on the process below 1MHz, 10kHz or 1kHz). There exist several theories about the origin of this noise which is difficult to measure due to the low frequency.

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5 Noise Applications

Consider the multi-channel communication system of Fig. 5.1. For each channel, an antenna is connected to an amplifier (receiver) through a lossy transmission line. The beam-former multiplies the signal of each channel by a complex weight and sums up all the weighted signals. In this section we shall develop expressions for the noise power at the output of each stage of the system. This will include the contribution of the noise percepted by the antenna, the absorption noise of the transmission line, the thermal noise of the receiver and the effect of the beam-former.

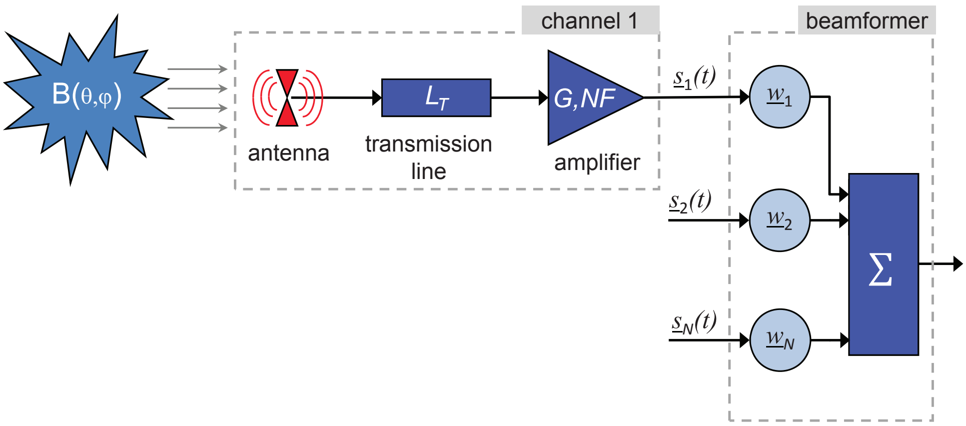

Figure 5.1: Microwave receiver system

5.1 Noise Performance of Cascaded Devices

It is convenient to develop a concise way to describe the amount of noise added to a signal passing through a device relative to a fixed standard. The *signal-to-noise ratio* is used to determine the degree of noise contamination of the signal. This leads to the definition of the *noise figure* as a figure-of-merit in comparisons between different devices. Using the noise figure it is easy to determine the noise performance of a number of cascaded devices, where each device contributes to the total noise at the output of the chain.

5.1.1 Noise Figure

Any real device always adds some noise so that the input signal-to-noise ratio is higher than the output signal-to-noise ratio. To measure the amount of degradation, we define a noise figure, NF, to be the ratio between the input and output signal-to-noise ratios, respectively:

$$NF = \frac{SNR_{in}}{SNR_{out}} \tag{5.1}$$

By definition, a fixed value for the input noise power is used when determining the noise figure of a device using (5.1). This noise power is equivalent to the thermal noise power provided by a resistor (as described in section 4.4.1) matched to the input and at a temperature of $T_0 = 290\,\mathrm{K}$.

The noise figure is commonly expressed in decibels:

$$NF^{\mathrm{dB}} = 10 \log_{10}(NF) \tag{5.2}$$

The noise figure of a perfect noise free device is unity (or 0 dB), and the introduction of additional noise causes the noise figure to be larger than unity, i.e. $NF^{\mathrm{dB}} > 0\,\mathrm{dB}$.

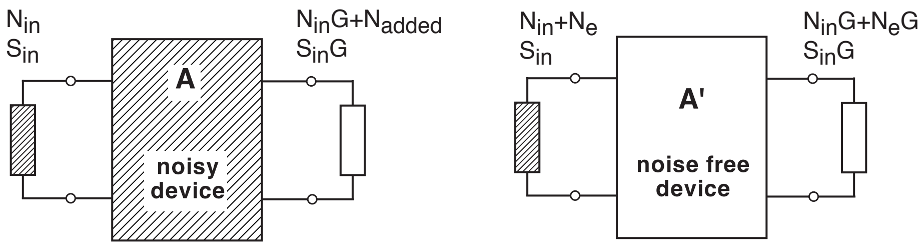

Figure 5.2: Noisy two port device and its equivalent model

Consider the two port device A as shown in Fig. 5.2 with the transfer function $H(f)$ and the equivalent noise bandwidth B_N. The gain of the device, defined as the ratio of the signal output power to the signal input power, is G. Thus the output signal power is[1] $S_{out} = S_{in}G$. The output noise power consists of the amplified input noise $N_{in}G$ in addition to the noise added by the device itself N_{added}, thus $N_{out} = N_{in}G + N_{added}$. To describe this added noise an equivalent noise free device A' will be assumed with a noise generator at its input such that the total output noise of A and A' are equal. As shown in Fig. 5.2 the output noise becomes $N_{out} = N_{in}G + N_eG$ and through (5.1) the noise figure can be represented as

$$NF = \frac{SNR_{in}}{SNR_{out}} = \frac{S_{in}/N_{in}}{S_{in}G/(N_{in}G + N_eG)} = 1 + \frac{N_e}{N_{in}} \tag{5.3}$$

[1] for simplicity we will write $S_{out,in}$ insted of $P_{S_{out,in}}$, and $N_{out,in}$ instead of $P_{N_{out,in}}$. These quantities denote the total power as described in section 4.2.4 equation (4.21)

The additional noise can be assumed to originate from an equivalent thermal noise source at temperature T_e thus $N_e = kT_eB_N$. By definition the input noise is $N_{in} = kT_oB_N$ and substituting into (5.3) gives

$$NF = 1 + \frac{kT_eB_N}{kT_oB_N} = 1 + \frac{T_e}{T_o} \tag{5.4}$$

It should be noted that the effective temperature T_e is only the *equivalent* physical temperature of a resistor that generates the same noise power as the device, the actual noise source might not be thermal. Nevertheless (5.4) gives a simple formula to calculate the effective temperature given the noise figure. The noise figure is useful for comparing different systems regarding their noise performance. The noise temperature on the other hand can be effectively used to calculate the actual amount of noise present in the system.

A better understanding of the meaning of the noise figure is possible by rewriting equation (5.3)

$$NF = 1 + \frac{N_e}{N_{in}} = \frac{N_{in} + N_e}{N_{in}} = \frac{G(N_{in} + N_e)}{GN_{in}} = \frac{N_{out}}{GN_{in}}. \tag{5.5}$$

From the above expression it is seen that the noise figure can be defined as the ratio of the total output noise to the total output noise of the noise free device, i.e.

$$NF = \frac{\text{total output noise}}{\text{total output noise of noise free device}} \tag{5.6}$$

Note: At the first glance (5.4) and (5.5) seem to be frequency independent since the noise figure is not dependent on the transfer function of the device. The justification would be that both noise and signal pass through the same device, so that $|H(f)|$ cancels out when forming the signal-to-noise ratio. This however is not correct since the noise generated within the device N_{added} will be frequency dependent in most cases, thus we should write $T_e(f)$ and keep in mind that F in (5.4) can at most be assumed constant within some frequency range. Based on (5.5) we can write an expression for the *band noise figure* $\overline{NF}$ which is frequency independent and gives the noise figure for the total frequency band

$$\overline{NF} = \frac{\int_0^\infty F(f)|H(f)|^2 N_{in} df}{\int_0^\infty |H(f)|^2 N_{in} df} = \frac{\int_0^\infty F(f)|H(f)|^2 df}{\int_0^\infty |H(f)|^2 df} = \frac{\int_0^\infty F(f)|H(f)|^2 df}{B_N|H(f_o)|^2} \tag{5.7}$$

In the above equation the gain G has been replaced by the square of the amplitude of the transfer function $|H(f)|^2$ which gives the relation between the input and output spectral power density (see (4.28)). The total output noise at each frequency is found as the product of the output noise from the noise free device $|H(f)|^2N_{in}$ times the noise figure. The last term in (5.7) makes use of the equivalent bandwidth from section 4.3.3.

5.1.2 Noise Figure in Cascaded Systems

In this section expressions for the noise figure for a combination of cascaded networks will be derived. Consider the cascaded two-port devices shown in Fig. 5.3.

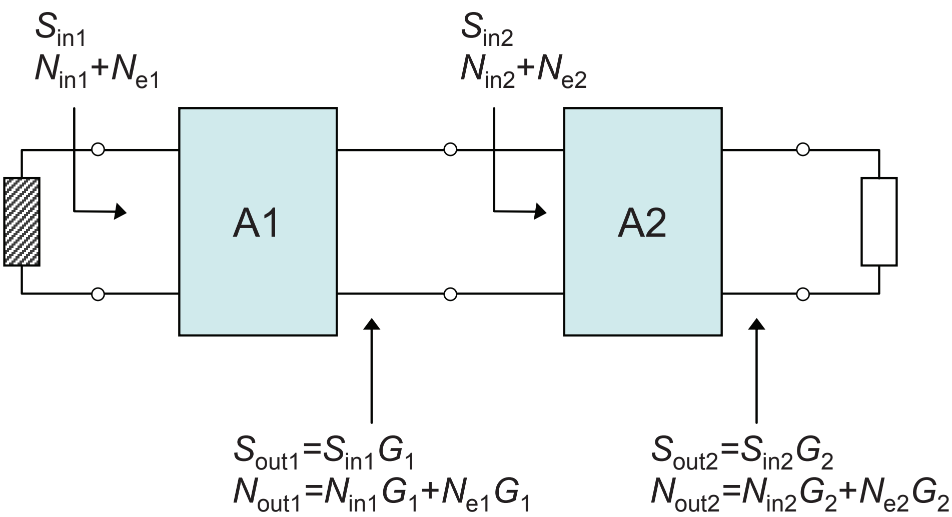

Figure 5.3: Equivalent model for the transmission of noise through a cascaded system

Using the definition of the noise figure from (5.5) and knowing that the total noise power output is $N_{out2} = G_1G_2(N_{in} + N_{e1}) + G_2N_{e2}$, the noise figure of the system NF_{12} will be:

$$NF_{12} = \frac{\text{total output noise}}{\text{total output noise of noise free device}} = \frac{G_1G_2(N_{in} + N_{e1}) + G_2N_{e2}}{N_{in}G_1G_2} \tag{5.8}$$

The effective noise temperature of the cascaded system can readily be obtained from NF_{12} and (5.4).

Equations (5.8) can be generalised to a series of N cascaded networks [Bundy, 1998]:

$$NF_{1N} = \frac{G_1G_2\cdots G_N(N_{in} + N_{e1}) + G_2\cdots G_N N_{e2} + \cdots + G_{N-1}G_nN_{e\,N-1} + G_NN_{e\,N}}{N_{in}G_1G_2\cdots G_N} \tag{5.9}$$

If the two-port networks are assumed to have identical input and output impedances, it can be shown that the minimum of the equivalent noise figure (or the equivalent temperature) can be reached if the networks are arranged with increasing noise figures of the individual stages.

The noise figure of cascaded networks (5.9) provides a simple and convenient way to evaluate the noise performance of a system. An important point to note however is that the noise figure assumes a perfect match between the input and output of all the networks and the cascaded structure. If this condition is not fullfiled, the easy-to-use equations can no longer be applied, the procedure is however straight forward and mainly involves the derivations already made.

Drill Problem 41 *A receiver for satellite transmissions at* 4 GHz *consists of an antenna preamplifier with a noise temperature of* 127 K *and a gain of* 20 dB*. This is followed by an amplifier with a noise figure of* 12 dB *and a gain of* 40 dB*. Compute the overall noise figure and equivalent noise temperature of the receiver. What would be the value of the noise figure if the order of the amplifier and preamplifier would be exchanged? Assume that the amplifiers are at a physical temperature of* 290 K*.*

5.2 Microwave Receiver Noise Temperature

Consider the system shown in Fig. 5.1. The voltage at the terminals of the input of the beam-former is

$$x(t) = a(\phi, \vartheta)s(t) + n(t) \tag{5.10}$$

where the first term represents the signal, while the second term is the noise voltage. Later the quantity of interest will be the power, which, assuming root-mean-square voltages, is written as

$$p_x = \langle x(t)x^*(t)\rangle = |a(\phi, \vartheta)|^2\langle|s(t)|^2\rangle + \langle|n(t)|^2\rangle \tag{5.11}$$

where $\langle\cdot\rangle$ denotes time-domain averaging and * the complex conjugate. In the above it has been assumed that the signal of interest is uncorrelated to the noise, thus $\langle s(t)n^*(t)\rangle = 0$, which is true for a non-multiplicative internal noise contribution.

In the following we investigate the various noise contributions, up to the input of the beam-former.

5.2.1 Antenna Noise Temperature

The noise at the output terminals of a lossless antenna, c.f. Fig. **??** is considered. All real antennas are directional antennas [Balanis, 1997], which is the property of radiating or receiving electromagnetic waves more effectively in some directions than in others. The directional properties of an antenna can be described through the radiation pattern (c.f. section 1.5). If we assume spherical coordinates, the radiation pattern $C(\theta, \psi)$ gives the ratio of the field strength in a given direction (θ, ψ) from the antenna to the maximum field strength. Whether we are dealing with a receiving or transmitting antenna is indifferent for the definition of $C(\theta,\psi)$. Since the radiation pattern is normalized to the maximum value, the ratio of power density to maximum power density is given through $C^2(\theta,\psi)$. When dealing with the received noise by the antenna, we will use the radiation pattern as a weighting function to combine the effect of the different noise sources with the directional properties of the antenna.

All noise sources contributing to the total noise power received by the antenna will be represented through their brightness B as described in section 4.4.2. As seen by the antenna, the brightness will be a function of direction (θ,ψ) referred to the antenna coordinates. The spectral density per unit area of the noise power is than given by:

$$s = \frac{1}{2}\iint_{4\pi} B(\theta,\psi)C^2(\theta,\psi)d\Omega \qquad [\mathrm{watt/m^2 \cdot Hz}] \tag{5.12}$$

In the above equation the noise is assumed to be unpolarised while the antenna is assumed to receive one single polarisation; this is the reason of the factor of $1/2$ since only half the noise power is received. Within the bandwidth Δf around the centre frequency f_o the power density is

$$S = \int_{f_o-\Delta f/2}^{f_o+\Delta f/2} \frac{1}{2}\iint_{4\pi} B(\theta,\psi)C^2(\theta,\psi)d\Omega\, df \qquad [\mathrm{watt/m^2}] \tag{5.13}$$

The available power at the antenna terminals is calculated using the *effective antenna area* A_r which by definition is the area when multiplied by the incident power density gives the power delivered to the load (c.f. section 1.5). The effective area and the radiating pattern are related through

$$A_r = \frac{\lambda^2}{\iint_{4\pi} C^2(\theta,\psi)d\Omega} \tag{5.14}$$

Using A_r and (5.13) to calculate the available noise power N_{ant} we get

$$N_{ant} = \frac{A_r}{2}\int_{f_o-\Delta f/2}^{f_o+\Delta f/2} \iint_{4\pi} B(\theta,\psi)C^2(\theta,\psi)d\Omega\, df \tag{5.15}$$

In the above equation the inner integral stands for the summation over all noise sources. The noise power depend strongly on the environment and look direction of the antenna. An antenna pointing towards empty space will have a very low noise temperature in the order of 3 K provided the side lobes aren't pointing towards a noisy environment. A narrow-beam antenna on the other hand with its beam directed toward the sun might encounter a noise temperature up to 300 000 K. The outer integral in (5.15) sums the noise power for the frequency band of interest and the multiplication factor of A_r transform the power density at the antenna to the available power at the antenna terminals, thus the effective area will also include the effect of antenna mismatch.

An expression as given by (5.15) is not practical when dealing with or comparing different receiving antennas. What is needed is a simple figure-of-merit such as the equivalent noise temperature of the antenna, which can be found by simplifying (5.15). A narrow bandwidth Δf is assumed such that the spectral brightness (4.46) can considered constant[2] over Δf . In addition, if the integration of the spectral brightness is compared to the integration of the frequency transfer function as described in section 4.3.3, an equivalent bandwidth B_N can be introduced which reduces the integration over Δf to the multiplication by the equivalent bandwidth at $f_0 = c/\lambda_0$, thus (5.15) becomes

$$\begin{aligned} N_{ant} &= \frac{A_r}{2}\int_{f_o-\Delta f/2}^{f_o+\Delta f/2}\iint_{4\pi}\frac{2kT_c(\theta,\psi)}{\lambda^2}C^2(\theta,\psi)d\Omega\, df \\ &= \frac{A_r}{2}\int_{f-\Delta f/2}^{f+\Delta f/2}\frac{2k}{\lambda^2}\iint_{4\pi}T_c(\theta,\psi)C^2(\theta,\psi)d\Omega\, df \\ &= A_r B_N \frac{k}{\lambda_0^2}\iint_{4\pi}T_c(\theta,\psi)C^2(\theta,\psi)d\Omega \end{aligned} \tag{5.16}$$

inserting (5.14) into the above equation gives

$$N_{ant} = k\left[\frac{\iint_{4\pi}T_c(\theta,\psi)C^2(\theta,\psi)d\Omega}{\iint_{4\pi}C^2(\theta,\psi)d\Omega}\right]B_N \tag{5.17}$$

Comparing the above equation with (4.42) for the thermal noise power from a resistor immediately suggest the definition of an equivalent antenna noise temperature T_{ant} of the form

$$T_{ant} = \frac{\iint_{4\pi}T_c(\theta,\psi)C^2(\theta,\psi)d\Omega}{\iint_{4\pi}C^2(\theta,\psi)d\Omega} \tag{5.18}$$

If the antenna is replaced by a resistance R_r at temperature T_{ant}, then according to (5.17) this resistance will generate the same noise power as by the antenna.

Consider a lossless microwave antenna placed inside an anechoic chamber maintained at a constant temperature T as illustrated in Fig. 5.4. The absorbing chamber completely encloses the antenna and is covered by absorbing materials which act as blackbody radiators. The power received by the antenna due to emission of the chamber is given by (5.17) with the brightness temperature $T_c(\theta,\psi)$ replaced by the constant temperature T seen by the antenna. Solving the integral we get

$$N_{ant} = k\left[\frac{\iint_{4\pi}TC^2(\theta,\psi)d\Omega}{\iint_{4\pi}C^2(\theta,\psi)d\Omega}\right]B_N = kTB_N \tag{5.19}$$

[2]note that the spectral brightness depends on frequency through $1/\lambda^2 = f^2/c^2$.

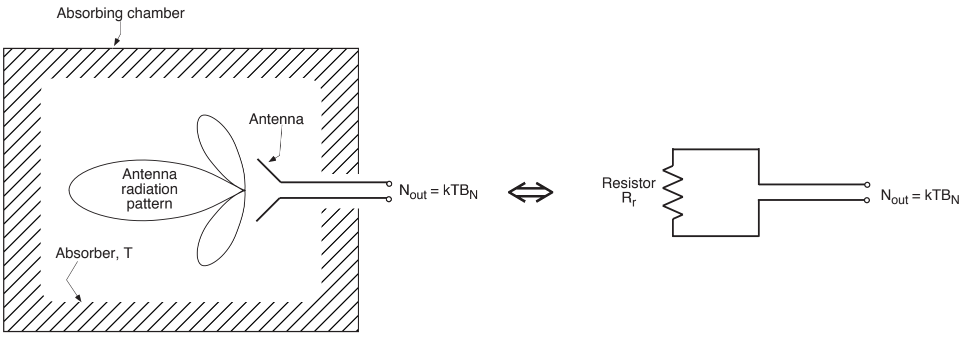

Figure 5.4: Noise power of an antenna placed inside an absorbing chamber

which is the same power available from a resistor at temperature T as given in section 4.4.1. From the standpoint of an ideal receiver of bandwidth B_N, the antenna connected to its input terminals is equivalent to a resistance R_r known as the *antenna radiation resistance*. Although in both cases the receiver is connected to a "resistor" in the case of the real resistor the noise power available at its output terminal is determined by the physical temperature of the resistor, while in the case of the antenna the power available is determined by the temperature of the blackbody enclosure, whos walls may be at any distance from the antenna. Moreover, the physical temperature of the antenna structure has no bearing on its output power as long as it is lossless.

An important perception from (5.19) is that the total noise power received by an antenna is *independent* of the radiation pattern of the antenna *if* the surrounding environment is assumed to have a constant brightness temperature.

Drill Problem 42 *A reflector antenna used for geostationary satellite receiving is positioned such that its main beam lies at* $37°$ *above the horizon. For simplicity the main beam is supposed to extend over* $\pm 1°$ *both in elevation and azimuth as shown in Fig. 5.5(a). The value of the radiating pattern outside the main beam is* $-32\,\mathrm{dB}$. *As shown in Fig. 5.5(b), the antenna 'sees' the sky with a brightness temperature of* $T_{c,sky} = 2\,\mathrm{K}$ *for* $\theta < 90°$, *the sun at* $T_{c,sun} = 8000\,\mathrm{K}$ *within a solid angle of* $\Omega_s = 0.5°$ *and the earth at* $T_{c,earth} = 300\,\mathrm{K}$ *for* $\theta > 90°$. *Calculate the antenna noise temperature with the sun outside the main beam of the antenna.*

5.2.2 Transmission Line

The noise temperature of a lossy transmission line can be calculated using the results of section 4.4.3. Consider the transmission line shown in Fig. 5.1. If the transmission line is at the physical temperature T_p and has an attenuation of $L_T = 1/G_T = P_{in}/P_{out}$, then the equivalent noise temperature T_{eT} at the input of the transmission line as given

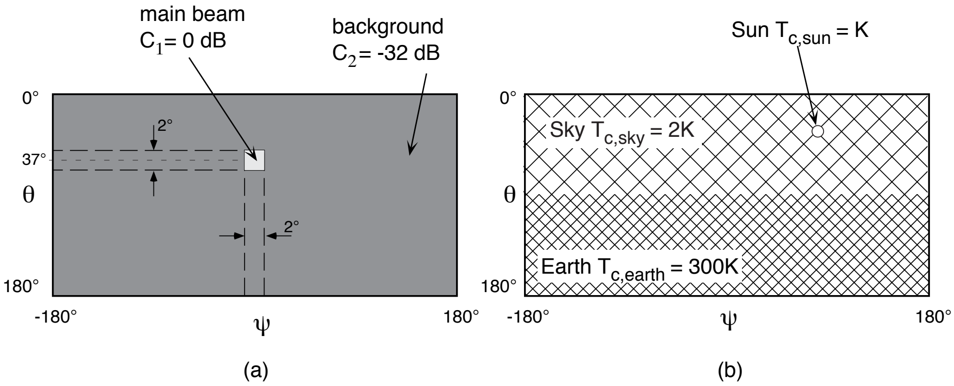

Figure 5.5: (a) Antenna pattern (not to scale) and (b) Brightness temperature

by (4.48) is

$$T_{eT} = T_p(L_T - 1) \tag{5.20}$$

which corresponds to a thermal noise power of $kT_{eT}B_N$. If the transmission line is connected to the terminals of an antenna having the noise temperature T_{ant}, the total noise power at the input of the transmission line becomes $N_{inT} = k(T_{eT} + T_{ant})B_N$. The noise power at the output of the transmission line is then

$$N_{outT} = \frac{N_{inT}}{L_T} = \frac{kT_p(L_T - 1)B_N}{L_T} + \frac{kT_{ant}B_N}{L_T} \tag{5.21}$$

and the noise temperature at the output becomes

$$T_{outT} = \frac{T_p(L_T - 1)}{L_T} + \frac{T_{ant}}{L_T} \tag{5.22}$$

which again corresponds to the noise power $kT_{outT}B_N$ at the output of the transmission line. The above results are only valid if the antenna is matched to the transmission line, thus assuming no reflected power at the transmission line terminal.

In practice, real antennas are not lossless devices. Part of the energy received (or transmitted) by the antenna is absorbed by the antenna material in the form of heat loss. This would require including the antenna losses when calculation the noise temperature T_{outT}. If the calculations are performed, it can easily be seen that the antenna losses might as well be included in the transmission line losses without altering the results, provided both the antenna and the transmission line are at the same physical, i.e. ambient, temperature T_p. This means, that equation (5.22) can be used with the antenna losses included in L_T.

5.2.3 Amplifier

Until now, the noise contribution of the antenna and the transmission line have been considered using the equivalent noise temperature. All quantities are referenced to the input of the amplifier, i.e. the noise power computed is at the input of the amplifier. Next, the aim is to get the noise power at the output of the amplifier, which is the noise power that goes into the beam-former. This is equal to the noise power $kT_{outT}B_N$ amplified by the gain of the G, and added to this we consider the contribution of the amplifier itself as given by its noise figure NF. The total output noise power[3] then becomes

$$\frac{\langle |n(t)|^2 \rangle}{R_o} = \mathsf{k} \left(\frac{T_{ant}G}{L_T} + \frac{T_p(L_T - 1)G}{L_T} + T_0(NF - 1)G \right) B_N \tag{5.23}$$

where the connection to the noise term in (5.11) is made.

5.2.4 Beam-Forming

Next the beam-forming is considered. To account for the multiple input signals, we add the index $_i$ where $i = 1 \ldots N$ and N is the total number of inputs (channels) to the beam-former. Then (5.10) for channel i becomes

$$x_i(t) = a_i(\phi, \vartheta)s(t) + n_i(t). \tag{5.24}$$

Note that in the above equation, the signal $s(t)$ is not indexed since it is the same to all input channels; this is consistent with taking the electric field strength at the antenna aperture to be $E(t)$ equally for all channels since it is attributed to a single point source at ϕ, ϑ.

Beamforming can then be described as the operation

$$y(t) = \sum_{i=1}^{N} w_i x_i(t) \tag{5.25}$$

where w_i specify the (complex) unitless weights. Using vector notation, thus writing $\mathbf{x} = [x_1(t), x_2(t), \ldots, x_N(t)]^T$, this becomes:

$$y(t) = \mathbf{w}^T\mathbf{x} = \mathbf{w}^T\mathbf{a}(\phi, \vartheta)s(t) + \mathbf{w}^T\mathbf{n} \tag{5.26}$$

It should be pointed out, that the term $\mathbf{w}^T\mathbf{a}(\phi, \vartheta)$ in the above equation can be used to explain the antenna array properties and effects; specifically, if normalized, this term represents the radiation pattern.

Now the power p_y at the output of the beam-former becomes

$$p_y = \langle y(t)y^*(t) \rangle = \langle (\mathbf{w}^T\mathbf{x})(\mathbf{w}^H\mathbf{x}^*) \rangle \tag{5.27}$$

[3]A receiver described through its gain and noise figure could replace the amplifier, as such the description is rather general

Evaluating the above using (5.26) and simplifying yields:

$$p_y = \mathbf{w}^H \mathbf{a}^*(\phi, \vartheta)\mathbf{a}^T(\phi, \vartheta)\mathbf{w}\langle s(t)s^*(t)\rangle + \mathbf{w}^H\langle \mathbf{u}^*\mathbf{u}^T\rangle\mathbf{w} \tag{5.28}$$

where the earlier assumption that the signal and noise are uncorrelated is maintained. If the noise of the individual channels is taken to be uncorrelated then $\langle n_i(t)n_j^*(t)\rangle = 0$ for $i \neq j$ and $\langle \mathbf{n}^*\mathbf{n}^T\rangle$ becomes a diagonal matrix.

Bibliography

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